2026

GENERAL MATHEMATICS

Level 3

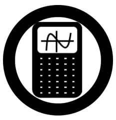
NETWORKS

General Mathematics: Level 3

GM3 - NETWORKS

By Jess Bertram

With sincere thanks to John Short and Rick Smith.

ICON:	MEANING:
	Worked example
	Complete with your teacher
	Try it yourself
	CAS Calculator can be used
	Tips / shortcuts

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GENERAL MATHEMATICS - LEVEL 3

GM3 NETWORKS

BY JESS BERTRAM

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Part 1 GRAPHS, PATHS, AND CYCLES

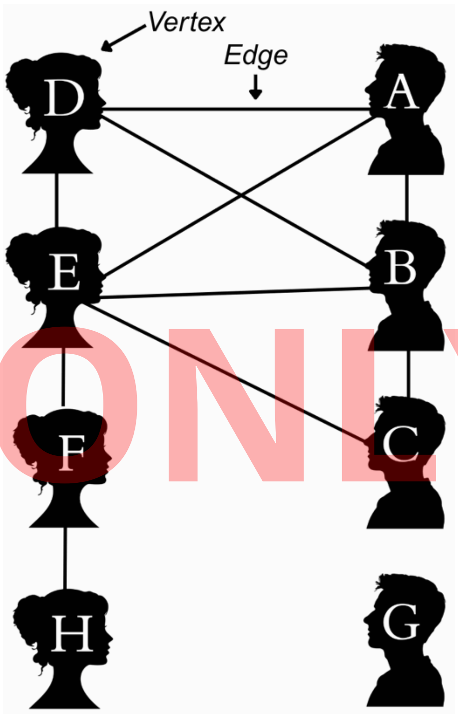
INTRODUCTION TO GRAPHS

A **graph** is a diagram that is used to represent a network.

The graph of a network consists of a set of points called **vertices**, which are connected by lines called **edges**.

The edges represent the connections or relationships between the vertices.

Example.



The Bertram siblings (Alex, Brad, Chris, Darrah, Ebony, Fallon, George and Hayley) have signed up for SnapChat.

The **graph** to the right shows the connection between each sibling.

Each sibling is represented by a **vertex**.

The connection between each sibling is represented by a line called an **edge**.

The line between Darrah (D) and Alex (A) shows that they are friends on SnapChat. Darrah is also friends with Brad (B) and Ebony (E).

Darrah has 3 connections total, so the **degree of the vertex** is 3.

With your teacher.

1. State the number of connections each sibling has. (degree of each vertex)

A= B= C= D= E= F= G= H=

2. How many total connections (edges) are there between all siblings?

3. Is George friends with any of his siblings on SnapChat?

4. Who is the most popular sibling? (Which vertex has the highest degree?)

5. How many vertices are there in the graph? What do they represent?

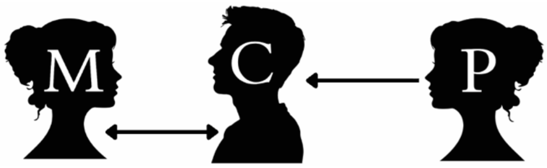
In the previous example, there were no arrows on the ends of the edges, so it is assumed the edges go both ways. Darrah is friends with Alex on SnapChat, and Alex is also friends with Darrah.

When an edge only goes in one direction it is represented by an arrow and called an **arc**. A graph with arcs in it is known as a **directed graph** or **digraph**.

Example.

The Bertram's are off to a dance. Their mother is trying to marry them off but can't keep track of who likes who. She knows two girls (Molly and Piper) like her son Chris, but Chris only likes Molly.

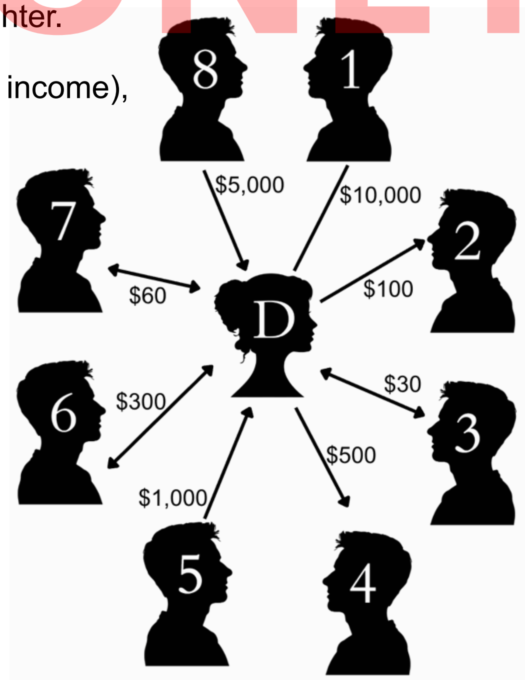
She draws a **directed graph** or **digraph** to represent the current situation.



Her daughter Darrah is proving quite popular and has a few marriage prospects. Some Darrah likes, some she doesn't.

Mumma Bertram draws a network digraph showing each potential match and their annual income to help make the decision with her daughter.

When the edges are assigned weights (such as annual income), the graph is called a **weighted graph**.



With your teacher.

1. How many men in the digraph is Darrah interested in marrying?

2. How many men are interested in marrying Darrah?

3. If Darrah wants to marry someone she likes, but also wants them to make over $100 a year, who should she choose?

With your teacher.

1. Construct a graph that could represent the following networks.
   a. Four current tennis champions meet in a ‘round robin’ tournament. (In a round robin, everyone plays everyone else).

   b. A network of roads connects five towns:
   Roads connect Southbridge directly to Franklin, Everton and Bell.
   There are also roads from Franklin to Everton and from Everton to Bell.
   There is an alternative direct route from Bell and Franklin that does not pass through Everton.
   A road connects Bell to Colt Ridge.

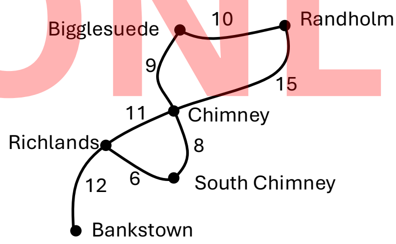
   c. Alan, Barry, Carrie are all friends. Alan also is friends with David and Elsa. Barry is friends with Elsa but does not know David.
      i. Draw a graph of the original friendship network.
      ii. How could you adjust the friendship network if Barry considers himself a friend of Carrie, but Carrie does not consider herself a friend of Barry?

Try it yourself (INTRO TO GRAPHS) *Answers page 77*

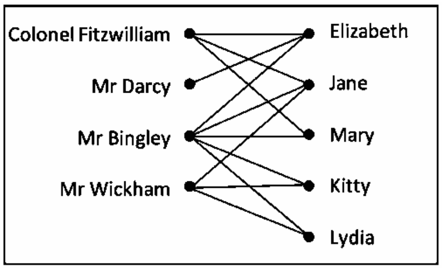
1. Draw a network graph for each of the following.
   a. Network 1.
      - A is connected with B and C.
      - B and C are also connected to D.
      - D is also connected to E.

   b. Network 2.
      - A is connected to B.
      - B is connected to C.
      - C is connected to D.
      - D is connected to A.
      - A and C are connected.
      - B and D are connected.
      - D is connected to E.

2. The graph shows a road network connecting towns.
   a. How many edges and how many vertices are used to draw the graph?
   b. Find the distance between Bankstown and Randholm via South Chimney and Bigglesuede.
   c. Find the distance between Bankstown and Randholm via the shortest possible route.



3. The network shows; ‘who danced with who’.
   a. Who did Elizabeth dance with?
   b. Who danced with the most partners?
   c. Who danced with the least?



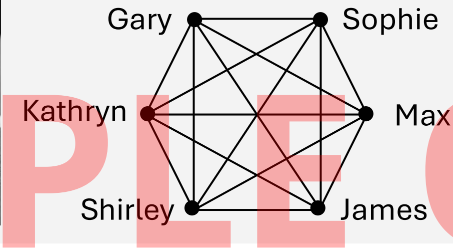
4. Construct a graph to represent the following friendship network:
   - Jenny is friends with Kate, Lenny and Maddy
   - Kate also counts Maddy as a friend but does not know Lenny. She is the only one to be a friend of Neil.
   - Maddy , Lenny and Oli are all friends.

Try it yourself (INTRO TO GRAPHS cont) *Ans pg77*

5. Construct a directed graph to represent a food web which includes the following: Kookaburra, mouse, grass, garden vegetables, possum, rabbit, dingo.
Why is a directed graph appropriate in this instance?

6. Consider the states of Australia (including ACT and NT).
    a. Construct a graph in which the vertices are the states. Join vertices with an edge if states share a common border.
    b. Which state is 'most connected'?

7. A tennis club organises a 'round robin' competition between six players. Each player plays one game against every other player.

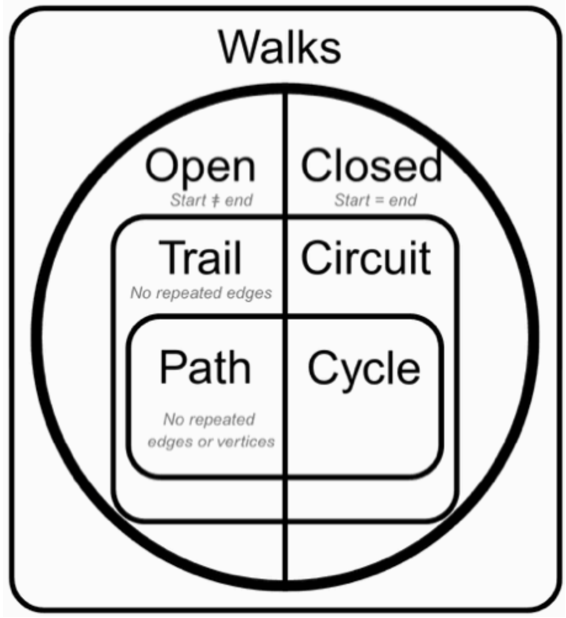


    a. What are the vertices of the graph?
    b. What do the edges represent?
    c. What is the significance of the degree of each Vertex?
    d. How many matches are played in total?
    e. Redraw the graph as a directed graph given the information that:
        - Kathryn won 5 matches.
        - Max beat Gary.
        - James won 4 matches
        - Sophie won 2 matches but lost to Max.
        - Shirley lost all her matches.
    f. Order the six players.

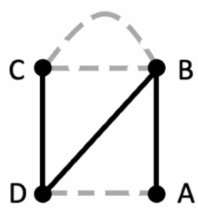
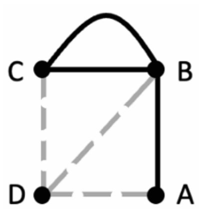
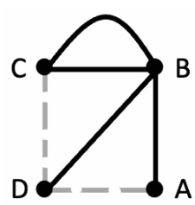
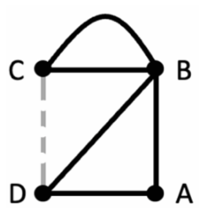
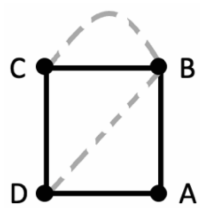
INTRO TO WALKS, TRAILS AND PATHS

When we move through a network, we can travel along its edges from one vertex to another.

Different terminology helps describe how we move through the network and whether we repeat edges or vertices.



Term		*Definition (examples below)*
Walk	→	A sequence of connected vertices *(all examples)*
Trail	→	A walk with no edges repeated *(Example 1 and 5)*
		A **circuit** is a closed trail *(Example 5)*
Path	→	A walk with no edges or vertices repeated *(Example 1 and 5)*
		A **cycle** is a closed path *(Example 5)*
Open	→	Starts and finishes at different vertices *(Example 1 and 3)*
Closed	→	Starts and finishes at the same vertex *(Example 2, 4, and 5)*

	Example 1	*Example 2*	*Example 3*	*Example 4*	*Example 5*
	ABDC	*ABCBA*	*ABCBD*	*ABCBDA*	*ABCDA*
					
Walk?	✓	✓	✓	✓	✓
Trail?	✓	✗	✗	✗	✓
Path?	✓	✗	✗	✗	✓
Open?	✓	✗	✓	✗	✗
Closed?	✗	✓	✗	✓	✓
Circuit?	✗	✗	✗	✗	✓
Cycle?	✗	✗	✗	✗	✓

EURLERIAN TRAILS

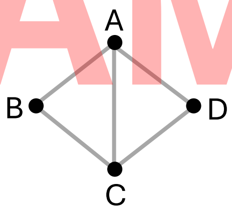
Some networks have special trails that let you travel along every edge exactly once. These are called Eulerian trails.

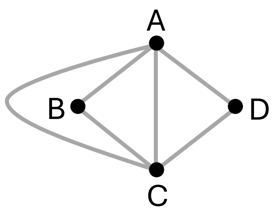
Eulerian and semi-Eulerian networks are used to model real-world problems like street-sweeping routes, garbage collection runs, and postal delivery paths — where each road (edge) must be covered once, without unnecessary repetition.

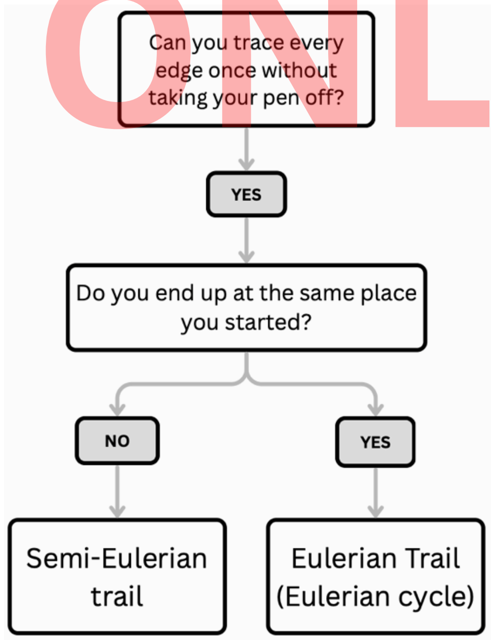
Is it a Eulerian trail?
If you can start at one vertex, trace every edge once (and only once) without lifting your pen off the graph, then return to the starting vertex, it's a Eulerian trail. If you can trace every edge once, but cannot return to the starting vertex, it is known as a **semi-Eulerian trail.**

With your teacher.

For the graphs below, see if you can trace a Eulerian or semi-Eulerian trail.







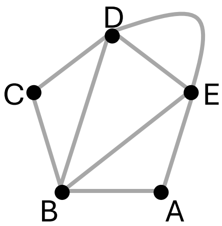
It is possible, by looking at the degree of the vertices, to determine if an undirected graph is Eulerian or semi-Eulerian without having to trace the edges.

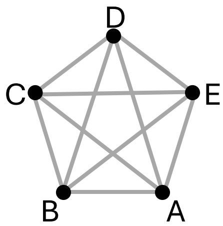
Eulerian trail	Semi-Eulerian trail
Has all even vertices.	Has exactly two vertices of odd degree.
Can start at any vertex.	Must start and finish at the odd degrees

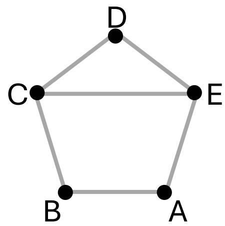
With your teacher.

For the following graphs determine if Eulerian or semi-Eulerian trails are possible.

Use both the tracing method and degree of vertex method.

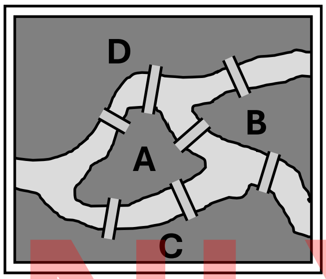






Worked example.

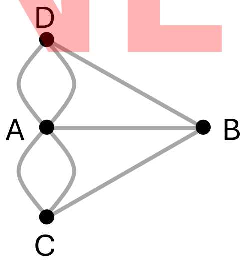
In the 1700s, the city of Königsberg (now Kaliningrad, Russia) was built around the Pregel River, which split the city into four land areas connected by seven bridges.



The people of Königsberg wondered:

> *"Is it possible to take a walk through the city that crosses each bridge once and only once, and return to where you started?"*

No matter how they tried, no one could find a route that worked.



In 1736, Leonhard Euler tackled the problem in a brand-new way.

He didn't focus on the exact map — instead, he simplified it:

- Each land area became a vertex.
- Each bridge became an edge connecting those vertices.

Euler proved mathematically that such a walk was impossible.

He discovered that to cross every edge once and end up where you started:

- Every vertex must have an even degree
- In Königsberg, all four land areas had an odd number of bridges — so it was impossible to create an Eulerian trail or circuit.

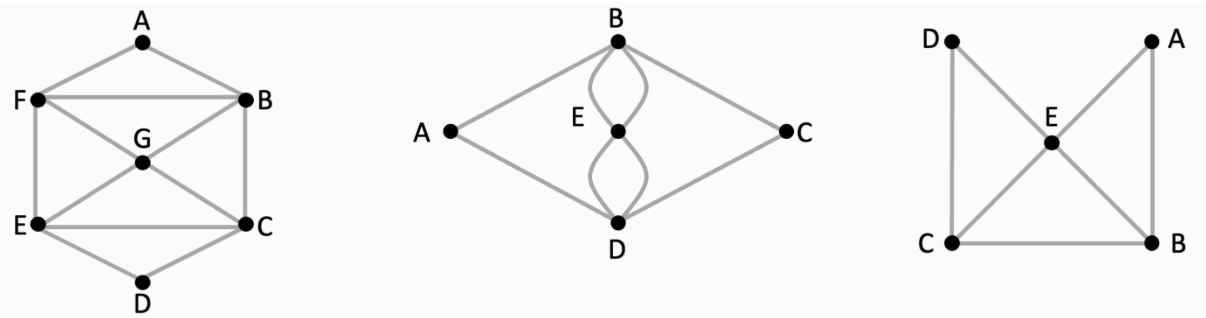
With your teacher.

Is it possible to tour the Konigsberg bridges – crossing each just once but completing your trail at a different point to where you started? Why (not)?

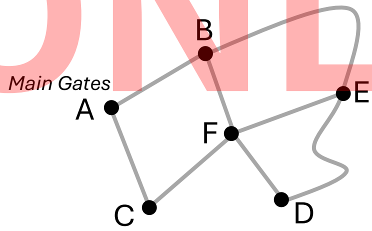
If you could destroy one bridge to make the trail work which one would you choose?

With your teacher.

1. Predict whether it is possible to complete an Eulerian trail or semi-Eulerian trail in each of the following graphs. If it is possible, find the trail.



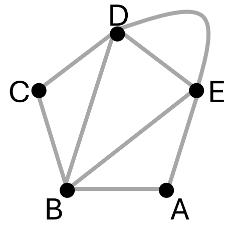
2. The graph shows the network of paths at a botanic garden.
   a. Would it be possible to walk the gardens, starting and finishing at the main gates, walking on every path and finishing back at the main gates. Why (not)?



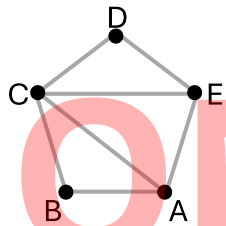
   b. Can you add an extra path so that such a walk is possible? Detail a possible walk with the new path in place.

Try it yourself (EULERIAN TRAILS) *Ans pg78*

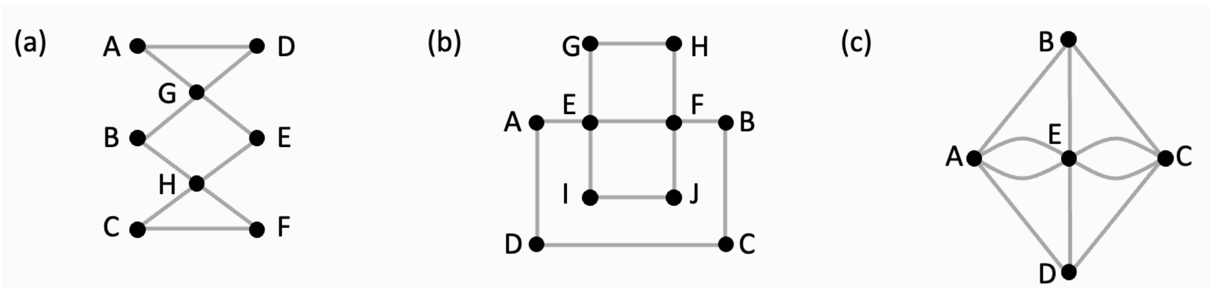
1. Using the graph presented give an example of:
    a. an open walk.
    b. an open walk that is not a trail.
    c. a closed walk that is not a trail.
    d. an open trail.
    e. an open trail that is not a path.
    f. a closed trail that is not a path.
    g. an open path.
    h. a closed path or cycle.



2. The statements refer to the graph shown. Mark each as true (T) or false (F).
    a. The network is connected.
    b. AB is a bridge.
    c. ABCD is an open walk.
    d. ABCD is an open path.
    e. CDEC is an open trail.
    f. CDEC is a closed trail.
    g. CDEC is a closed path.
    h. CDECA is a closed trail.
    i. CDECA is a open trail.
    j. CDECA is a open path.
    k. ABCDEA is a closed walk, closed trail and closed path.



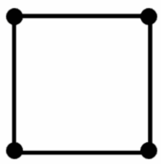
3. Find, if possible, an Eulerian trail for each of the following graphs (starting at A). If there is no Eulerian trail then find, if possible, a semi-Eulerian trail (starting at any vertex).



Try it yourself (EULERIAN TRAILS cont) *Ans pg78*

4. For each of the following graphs
   a. Find an Eulerian trail (if possible)
   b. Find the degree of each vertex
   c. Record the total number of vertices that have an odd degree and the total number of vertices with an even degree.
   d. Write a conclusion on when it is possible to find an Eulerian trail in a graph.

a.

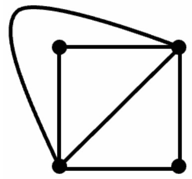


Eulerian trail (yes / no)

# of odd vertices ____

# of even vertices ___

b.

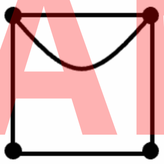


Eulerian trail (yes / no)

# of odd vertices ____

# of even vertices ___

c.

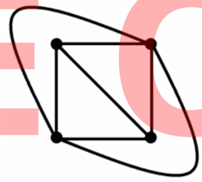


Eulerian trail (yes / no)

# of odd vertices ____

# of even vertices ___

d.

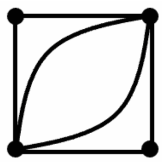


Eulerian trail (yes / no)

# of odd vertices ____

# of even vertices ___

e.

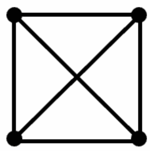


Eulerian trail (yes / no)

# of odd vertices ____

# of even vertices ___

f.

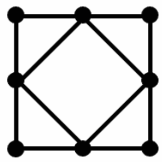


Eulerian trail (yes / no)

# of odd vertices ____

# of even vertices ___

g.

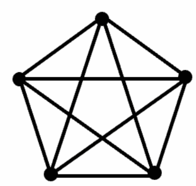


Eulerian trail (yes / no)

# of odd vertices ____

# of even vertices ___

h.



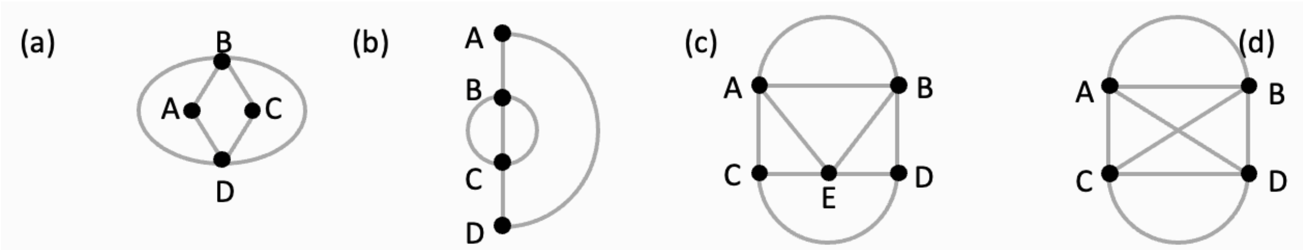
Eulerian trail (yes / no)

# of odd vertices ____

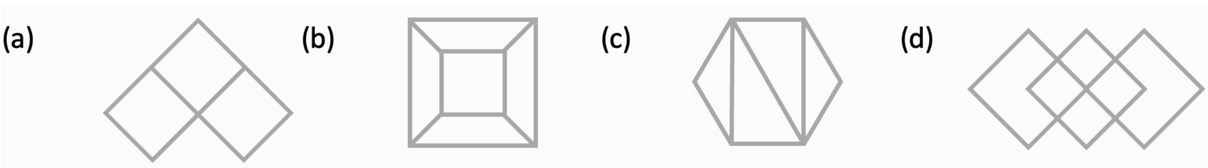
# of even vertices ___

Try it yourself (EULERIAN TRAILS cont) *Ans pg78*

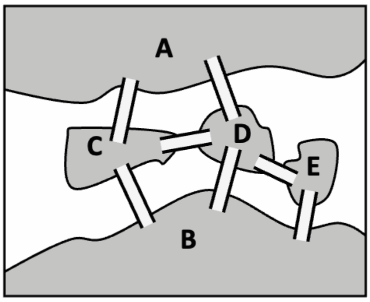
5. Predict whether it is possible to complete an Eulerian trail or semi-Eulerian trail in each of the following – before you find it.



6. Which of the following designs can be drawn without lifting pen from paper? Which can be drawn without lifting pen from paper and finishing at the starting point?

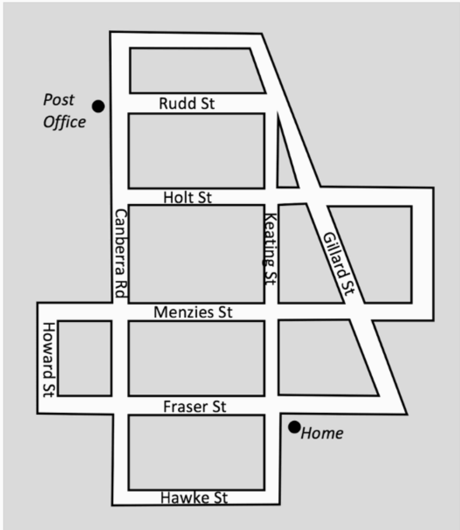


7. A city, (a little bit like Konnisberg), is situated on a river that is studded with small islands. The city is connected by a series of bridges as shown.
   a. Is it possible to take a circuit walk through the city crossing every bridge just once and finishing at your starting point? (How can you tell?)
   b. Would it be possible to take the walk crossing every bridge just once but finishing at a different point to where you started? How can you tell? Detail the route.
   c. Would it be possible to make a circuit walk if you destroyed one bridge. Which would it be? Detail a circuit starting from A.
   d. Would it be possible to make a circuit walk by adding an additional bridge? Where? Detail a circuit starting from A using the extra bridge.

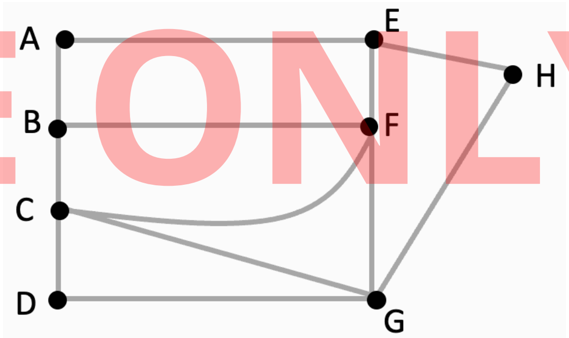


Try it yourself (EULERIAN TRAILS cont) *Ans pg78*

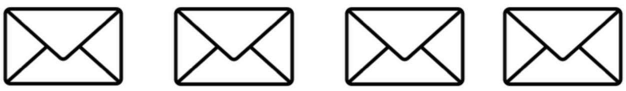
8. A postman delivers letters to all houses on roads in a city network.
   a. Is there a route that can be used so that the postman can start and finish at the post office which does not involve retracing steps? How could you tell?
   b. If, instead, the postman is allowed to return to his home after finishing his deliveries, is there a route which does not involve retracing steps?
   c. Where would the postman need to live for this to be possible?



9. A road-sweeper must clean all roads shown before returning to its depot at A.
   a. Is it possible to sweep all the roads and return to the depot without travelling on a road more than once? Why (not?)
   b. Can a route be found which involves no retracing if there is no requirement to start and finish the sweeping at the depot? Where does it start and finish? Detail the route.



10. A classic puzzle involving Eulerian trails is:
   "Can you draw an envelope without taking your pen from the paper?"
   a. Can it be done?
   b. Is this an example of an Eulerian trail or a semi-Eulerian trail?
   c. How could you prove it without even drawing the diagram?



Shortest path with weighted graphs.

Google Maps and other navigation systems use the same networks ideas you are learning about — then apply them to thousands (or millions) of points at once.

- Every intersection or location is treated as a vertex.
- Every road is an edge connecting those vertices.
- Each edge has a weight — for example, the distance or estimated travel time.

When you ask for directions, digital maps search through their enormous networks to find the shortest or fastest path from your starting point to your destination.

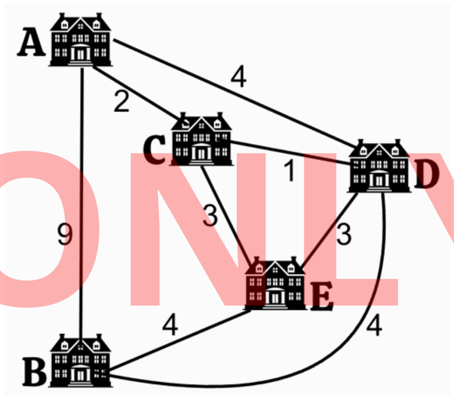
Google maps use algorithms such as Dijkstra's to find the shortest path between two vertices. In this course you will learn the basic concepts through trial and error.

Worked example.

The Bertram siblings have all moved house again. Alex, Brad, Chris, Darrah and Ebony are all within walking distance of each other.

Find the shortest route from Alex's house to Brad's.

(The numbers represent kms)



Using trial and error.

Write down every possibly path, then add up the total distance in kilometres.

Path	*Working*	Total kms
A → B	*9*	9
A → C → E →B	*2 + 3 + 4*	9
A → C → D → E → B	*2 + 1 + 3 + 4*	10
A → C → D → B	*2 + 1 + 4*	7
A → D → B	*4 + 4*	8
A → D → E → B	*4 + 3 + 4*	11

The shortest route is A → C → D → B, which is 7 kilometres.

It should be the quickest, as long as you don't stop for a cuppa at each house!

The 'Chinese Postman' problem

Involves finding the **shortest possible route** that travels along **every edge of a network at least once** and returns to the starting point.

It's used to plan efficient routes for tasks like mail delivery, garbage collection, or street sweeping — where every road must be covered with minimal total distance.

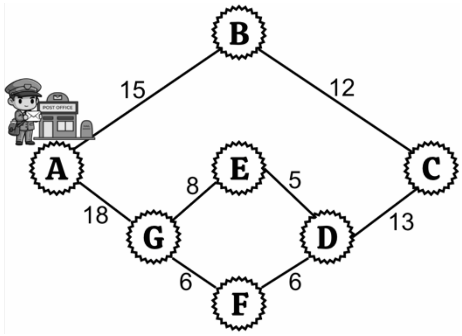
Worked example.

Postman Pete must travel every street in a town, delivering mail and then return to the post office (A). How can he do this while travelling the shortest possible distance?

Step 1: Find the degrees of the vertices.

If all vertices are even, then the network is Eulerian. Postman Pete could travel every street once and return to the start. No retracing needed.

→ Vertices D and G are odd, so no Eulerian network.



Step 2: Find the shortest option to retrace edges between any odd vertices.

The network is not Eulerian, so Postman Pete must retrace some edges.

As D and G are of odd degree, he will need to travel between D and G twice.

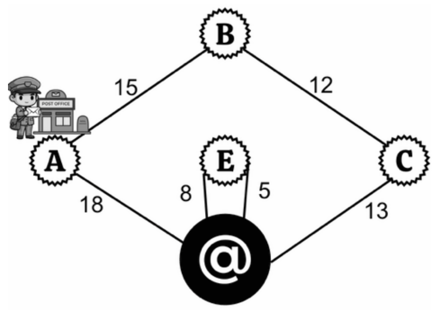
The shortest option between D and G is via F.

If we have to travel D → F → G → F → D it will take 24kms total.

We can replace this double up (D → F → G → F → D) with a random symbol to simplify. In this example we will use @.

Step 4: Use trial and error to find shortest route.

If we replace the double up edges with @, there is only one route which Postman Pete can complete clockwise or anticlockwise.



Option 1 (clockwise)	Option 2 (anticlockwise)
A → B → C → @ → E → @ → A	A → @ → E → @ → C → B → A
A (15) B (12) C (13) @ (5) E (8) @ (18) A = 71km	A (18) @ (8) E (5) @ (13) C (12) B (15) A = 71km
71 kms + 24kms (from double up) = **95 kms total**	71 kms + 24kms (from double up) = **95 kms total**

Step 5: Rewrite the route without the @ symbol.

A → B → C → D → F → G → F → D → E → G → A   Shortest possible path

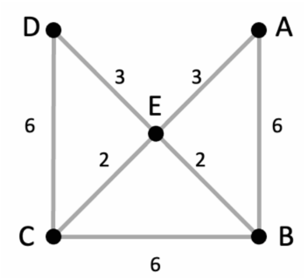
The shortest possible distance whilst still travelling every street is 95kms.

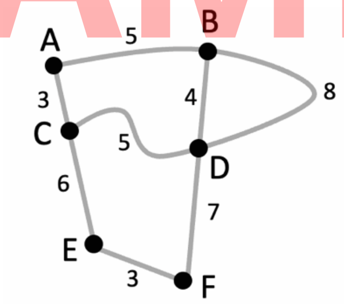
With your teacher.

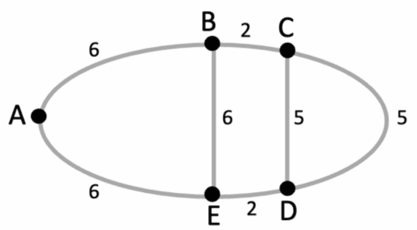
Postman Pete thinks 95km is a bit too far to travel every day. He is an excellent postman and has been offered jobs at three different cities. He wonders if one of those cities would be less travel and hence less work.

Each city is shown as a network in the graphs below.

For each city, determine the shortest route and minimum distance travelled in kilometres if Pete needs to start and finish at the post office (A).



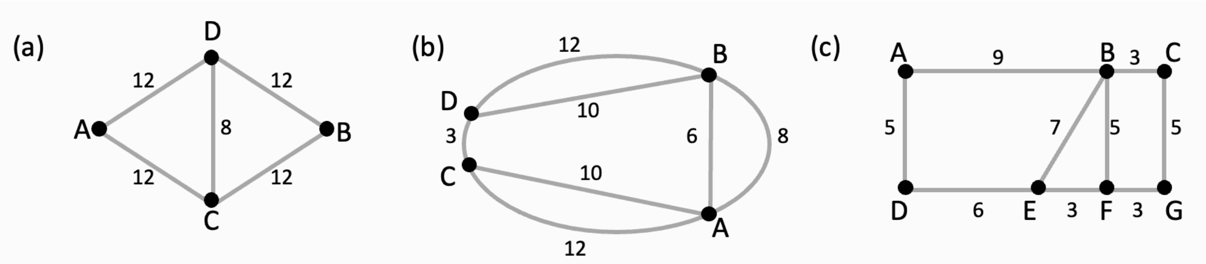




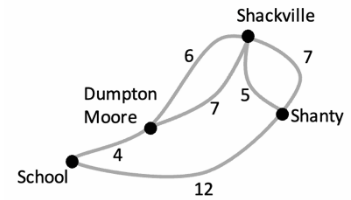
Which city would be the least amount of travel for Postman Pete?

Try it yourself (WEIGHTED EULERIAN TRAILS) *Ans pg78*

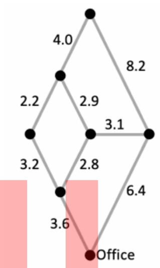
1. Solve the 'Chinese Postman Problem' for the following networks. Also determine the minimum distance travelled (in km). (Start and finish at A)



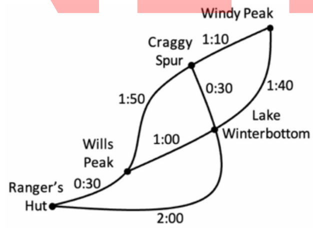
2. A school bus must pick up all the students living in a district and return to school. What is the shortest distance that the bus could cover to travel every road? (Distances in kms)



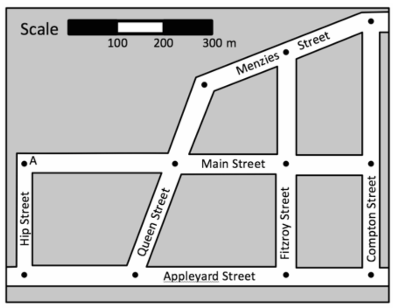
3. A National Park ranger regularly walks all the tracks in the park to ensure that they are in good condition and free of debris and litter. The map shows the track network weighted with walk times. What is the fastest time that the ranger could inspect all the tracks and return to his hut?



4. A politician doorknocks houses in her electorate, part of which is represented by the graph shown. (Distances in km.) What is the shortest distance that the politician could cover if she wished to cover all houses and start and finish at her office?



5. Simon delivers 'junk mail' in a metropolitan area as detailed in the map. He parks his car at point A and then travels on foot. Mark on the map the route for minimising the total distance and use the scale of the map to find its length.

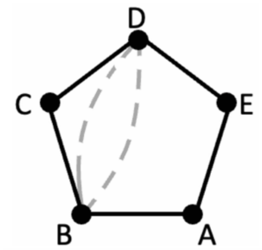


HAMILTONIAN PATHS

A **Hamiltonian path** is a path that includes every vertex exactly once.

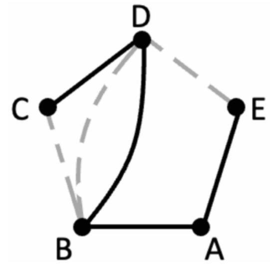
A **Hamiltonian cycle** is a path that includes every vertex and ends at the vertex it started at. (ie. A closed Hamiltonian path.)

Examples.



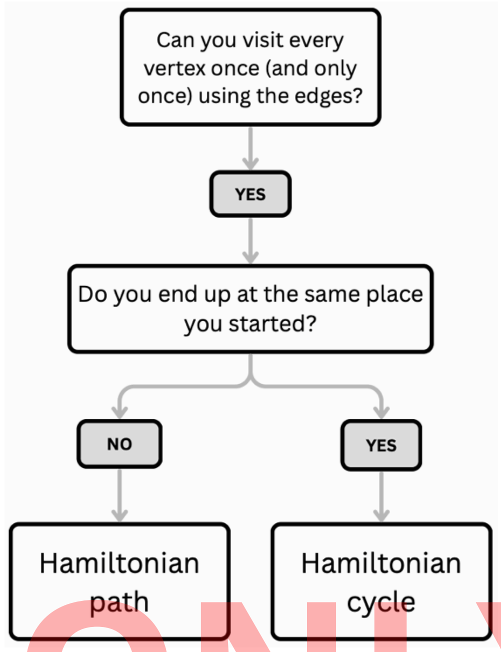
$A \to B \to C \to D \to E \to A$

Hamiltonian cycle



$E \to A \to B \to D \to C$

Hamiltonian path

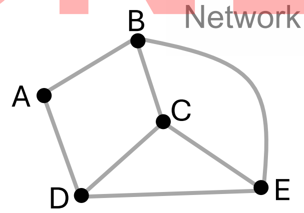


With your teacher.

1. For the network shown (network 1):
   a. Find a Hamiltonian cycle starting at A.

   b. Find a Hamiltonian path from A to C.

Network 1



2. For the network shown (network 2) find, if possible, a:
   a. Hamiltonian cycle starting and finishing at A.

   b. Eulerian trail starting and finishing at A.

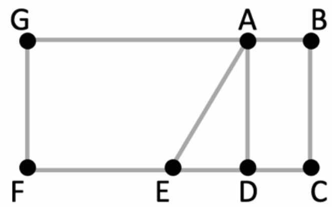
   c. Semi-Eulerian trail between any two vertices.

   d. Hamiltonian path from E to B.

   e. Hamiltonian path from G to B.

   f. Explain the difference between a Hamiltonian cycle and an Eulerian trail.

Network 2



The 'Travelling Salesman' Problem

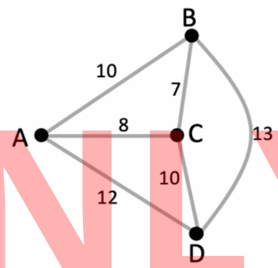
The Travelling Salesman Problem illustrates the use of Hamiltonian cycles in a real world context. The salesman must visit every town in his area before returning to his home. How should he route his journey so that his overall distance is minimised?

In other words: What is the shortest Hamiltonian cycle through the network?

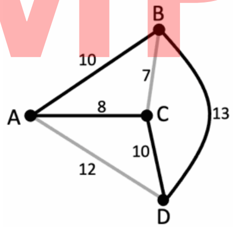
The 'travelling salesman' is one of the great unsolved problems of mathematics. There is no general algorithm for solving this type of question. The best way is to consider all possible Hamiltonian cycles, then select the shortest.

Worked example.

A salesman leaves his home at A. He must visit towns B, C and D before returning home. What route should he take if he wishes to minimise the total distance travelled?



Possible route 1

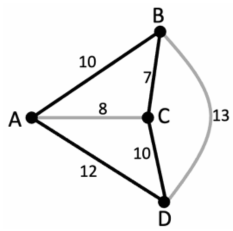


$A \to B \to D \to C \to A$

A (10) B (13) D (10) C (8) A

= 41 km

Possible route 2

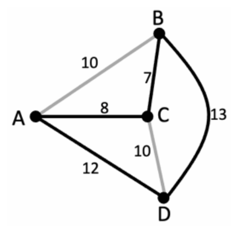


$A \to B \to C \to D \to A$

A (10) B (7) C (10) D (12) A

= 39 km

Possible route 3



$A \to C \to B \to D \to A$

A (8) C (7) B (13) D (12) A

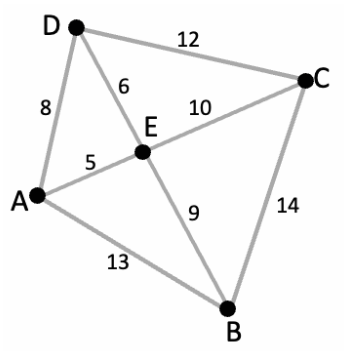
=40 km

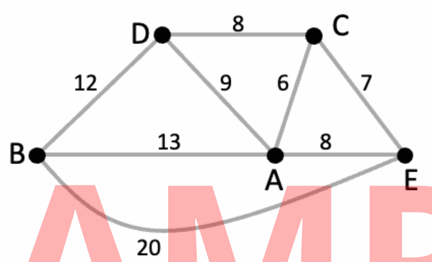
The best route would be $A \to B \to C \to D \to A$ with a total distance travelled of 39km

With your teacher.

Complete the Traveling Salesman problem for the following graphs:

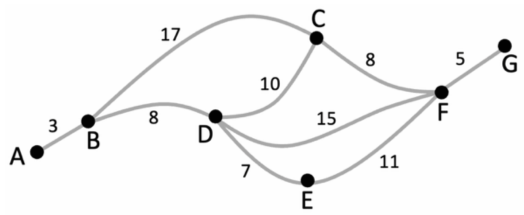
(The salesman leaves from and returns to A)





Worked example.

Find the shortest distance between A and G. (measurements are in km).



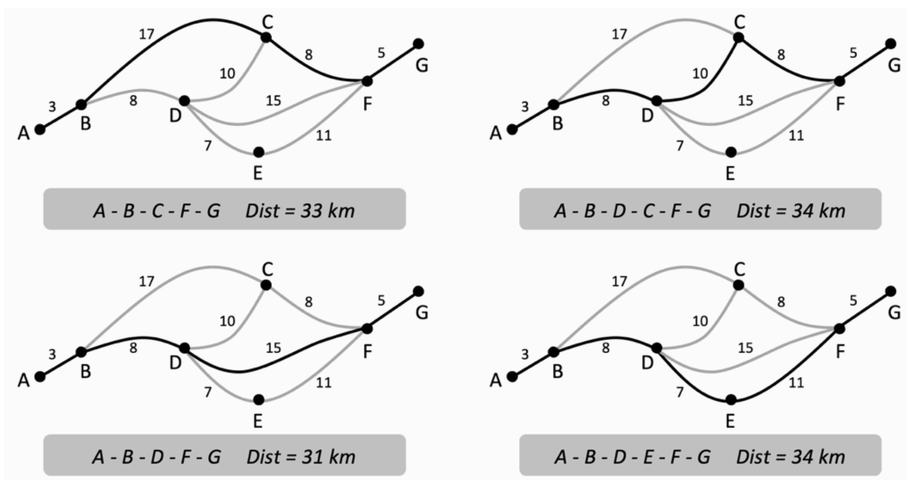
Some algorithms exist to help find the shortest path.

In this course we only cover trial and error.

(Can be called 'conducting an exhaustive search')

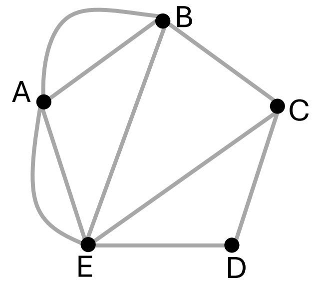
Route A – B – D – F – G

is the shortest (31km).

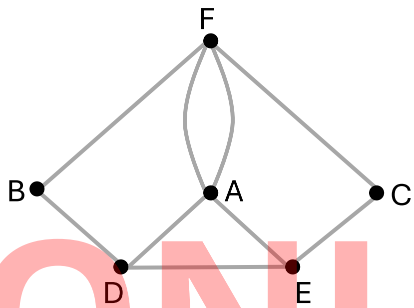


Try it yourself (HAMILTONIAN GRAPHS) *Ans pg78*

1. For the graph shown find, if possible, a:
    a. Hamiltonian cycle starting and finishing at A.
    b. Eulerian trail starting and finishing at A.
    c. semi-Eulerian trail between any two vertices.
    d. Hamiltonian path from D to C.
    e. Hamiltonian path from D to A.
    f. Hamiltonian path from D to B.
    g. Hamiltonian path from E to B.

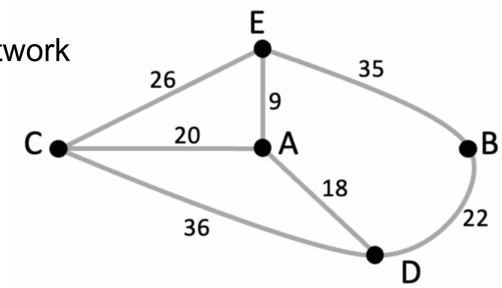


2. For the network shown find, if possible, a:
    a. Hamiltonian cycle starting and finishing at A.
    b. Eulerian trail starting and finishing at A.
    c. semi-Eulerian trail between any two vertices.
    d. Hamiltonian path from B to D.
    e. Hamiltonian path from B to F.
    f. Hamiltonian path from B to A.
    g. Hamiltonian path from D to C.



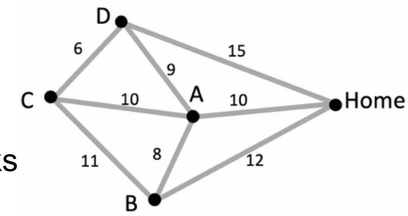
3. What type of path (Eulerian trail, semi-Eulerian trail, Hamiltonian cycle, or Hamiltonian path) might be involved in each of the following situations:
    a. A national parks ranger leaves the ranger station and walks all the tracks in the park before returning to the station.
    b. A couple aim to try every ride at a theme park. They must enter and exit the park through the main entry gates.
    c. Simon collects 'junk mail' from an agent and delivers it to houses on all the roads in a district finishing at his own home.
    d. A grounds-person must walk around an oval to eject all the in-ground sprinklers starting and finishing the journey at a water mains.
    e. A car rally starts and finishes at different locations. The organisers of the rally give competitors a list of places they must visit. The places may be visited in any order.

4. Solve the 'Travelling Salesman Problem' for the network shown. Start and finish at A. (Distances in km.)

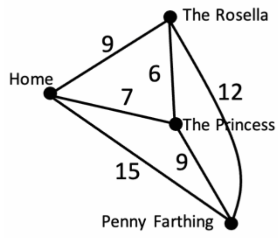


Try it yourself (HAMILTONIAN GRAPHS cont) *Ans pg78*

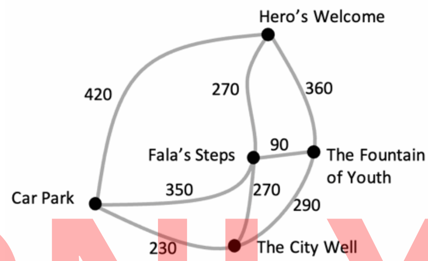
5. An ice-cream truck makes a regular tour of a city stopping at parks and playgrounds according to the network shown. Find the shortest circuit he can make which includes all parks and playgrounds before returning home. (distances in km)



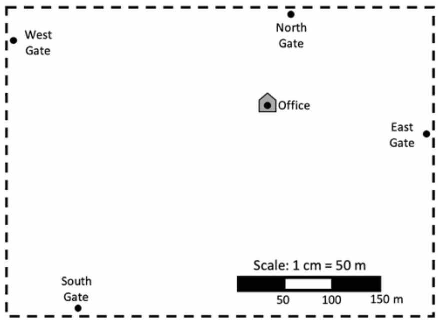
6. A caterer prepares cakes for cafés as shown in the network.
   a. Find as many Hamiltonian cycles as you can.
   b. Design a daily delivery route for the caterer that minimises the distance travelled. (Distances are in km.)



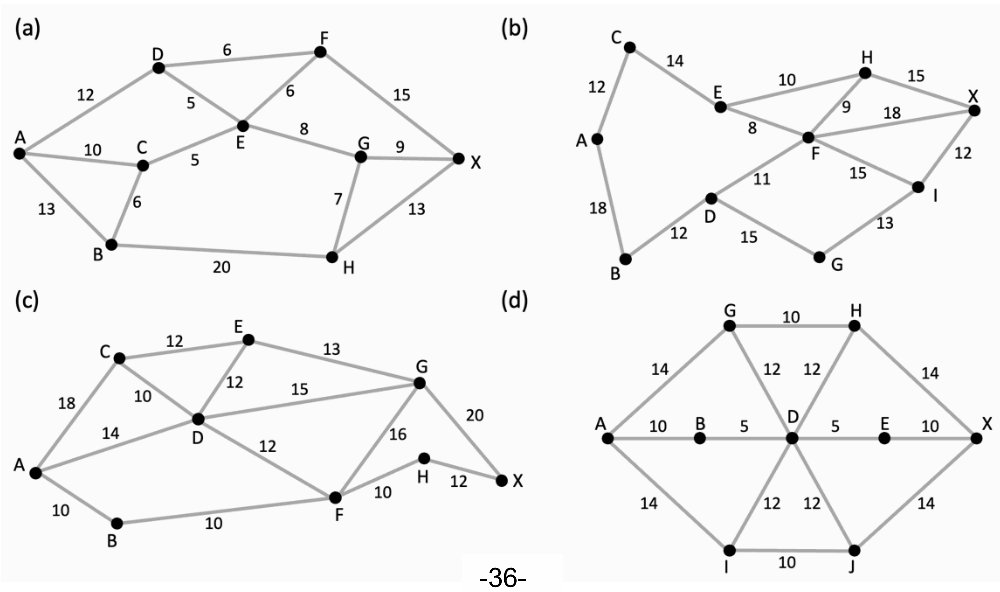
7. Visitors to the archaeological attractions at *The Lost World* leave their cars at a car park and proceed on foot using a network of walkways as shown on the diagram. (Distances are in m)
   a. List alternative Hamiltonian cycle that might be used.
   b. Which is the best route to view the attractions if you wished to see them all but minimise the total distance walked?



8. Every hour throughout the night a security guard makes a tour of a prison's four exits before returning to his surveillance office.
   a. What type of path is the guard using?
   b. List alternative routes that might be followed.
   c. Use the scale of the plan to determine the route that he should walk in order to cover minimum distance.
   d. What is the minimum distance involved?

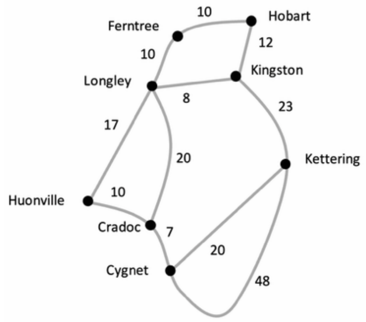


9. For each of the following graphs determine the shortest route between A and X.

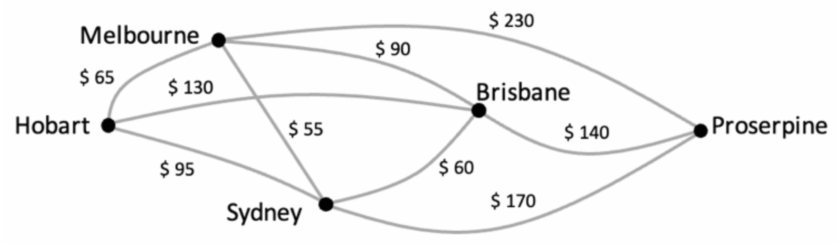


Try it yourself (HAMILTONIAN GRAPHS cont) *Ans pg78*

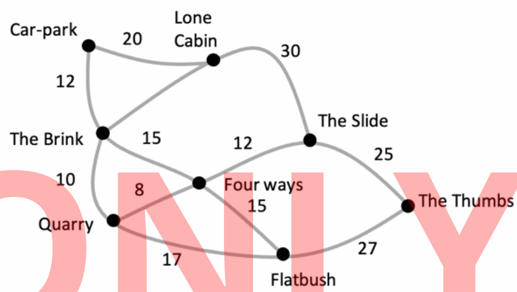
10. Find the shortest distance (in km) from Hobart to Cygnet. Is the shortest route always the quickest? Explain your answer.



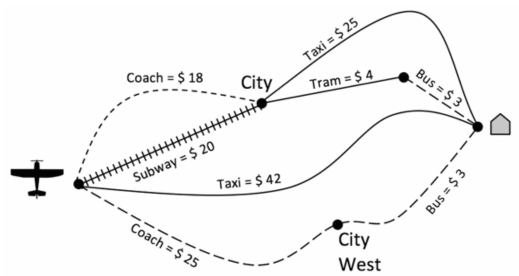
11. The network shows the cost of air flights between Hobart and Proserpine (Airlie Beach). What is the cheapest way to fly between Hobart and Proserpine?



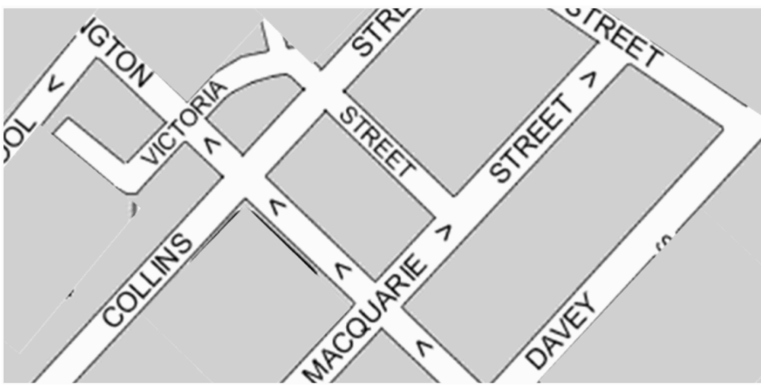
12. The graph depicts a network of mountain bike trails. The weights relate to the cycling times in minutes. Find the quickest route from the car park to 'The Thumbs'. How long will it take?

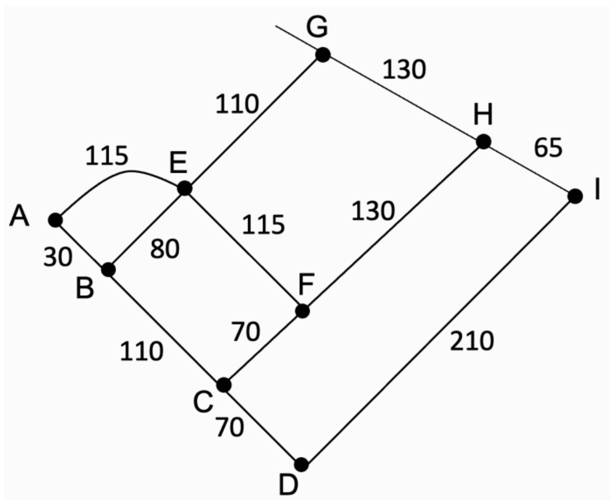


13. The graph shows the cost of transport between an airport and a suburban home. What is the cheapest way of getting from the airport to home?



14. A map of some Hobart streets and a matching network weighted with distances in metres is shown below. Find the shortest pedestrian route from A to I.





Part 3 FLOW NETWORKS

INTRODUCTION TO FLOW NETWORKS

In many real-life situations, something needs to be transferred through a system — water through pipes, vehicles through streets, data through a network, or power through electrical lines.

We can represent these systems using **flow networks**.

A flow network is a type of directed, weighted graph where:

- **Vertices** represent junctions, intersections, or stations.
- **Edges** represent pathways between them (like pipes or roads).
- The **weight** on each edge represents its **capacity** — the maximum amount that can flow through that path.

Key terms	
Source	The starting point of the flow (e.g. a water tank, oil depot, or data server). All edges *leave* this vertex.
Sink	The destination where flow ends (e.g. a water treatment plant or consumer). All edges *enter* this vertex.
Flow	The actual amount moving along each edge (which must not exceed its capacity).

Purpose of Flow Networks

We use flow networks to answer questions like:

- *What is the greatest possible flow from source to sink?*
- *Which pathways limit (bottleneck) the overall flow?*
- *How can we increase the total flow through the network most efficiently?*

These problems are solved using two approaches:

1. **By inspection:** Logical reasoning about which edges limit flow.
2. **Maximum-Flow Minimum-Cut Theorem:** A method that uses cuts through the network to find the theoretical maximum flow possible.

Maximum flow by inspection.

In a flow network, each edge has a capacity which tells us the maximum amount that can pass through it.

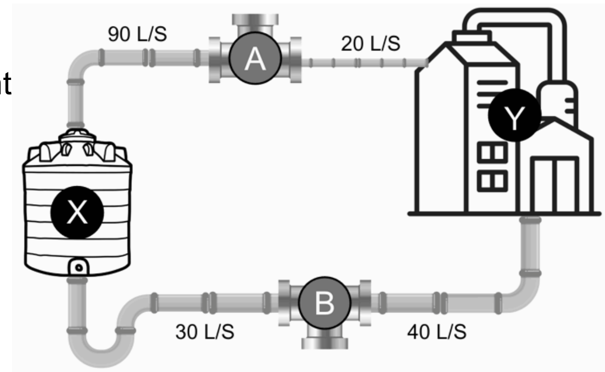
To find the maximum flow from the source to the sink, we can sometimes use logic alone — this is called finding the maximum flow by inspection.

The total flow through a network is limited by the smallest capacity on each possible route from the source to the sink. This edge acts like a *bottleneck* — it limits how much can flow through that entire path.

If there are several possible paths, the overall maximum flow is the sum of the flows through each path, as long as they don't share a bottleneck edge.

Steps to find maximum flow (by inspection)	
Step 1:	**Identify the source and sink.** These are the start and end points of the flow.
Step 2:	**List all possible paths** from source to sink.
Step 3:	**Find the bottleneck** for each path — the smallest capacity (weight) along that path.
Step 4:	**Add the flows** from each path together, provided the paths don't overlap. If paths share an edge, that edge's total flow can't exceed its capacity.

Worked example.



A water tank **(X)** pumps water to a treatment plant **(Y)** through two routes, A and B.

Each pipe has a maximum capacity, measured in litres of water per second (L/S).

X is the source as all pipes lead from it.

Y is the sink as all pipes lead to it.

Possible paths are path A (X → A → Y) and path B (X → B → Y)

The water tank can pump a maximum of 20 L/S through path A at a time.

The water tank can pump a maximum of 30 L/S through path B at a time.

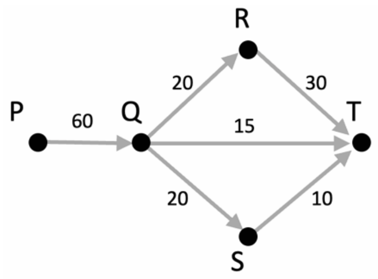
So, using both paths, the **total maximum flow from X to Y is 50 L/S.**

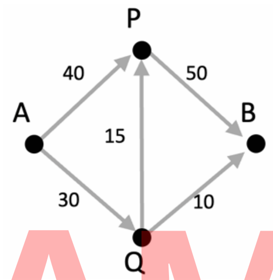
If we wanted to increase the total flow through the system, we would need to upgrade the smallest-capacity pipes — the ones that are restricting flow along each path.

With your teacher.

Identify the source and sink in each of the following networks.

Find 'by inspection' the maximum flow from source to sink.





Finding the maximum flow *by inspection* works fine for small networks.

But most real-life networks are not small – like planning water pipes for towns, motor vehicle traffic through roads and highways, or running data cables to reach homes.

When the networks get larger, the **Minimum cut – Maximum flow theorem** gives us proof that we've reached the true maximum.

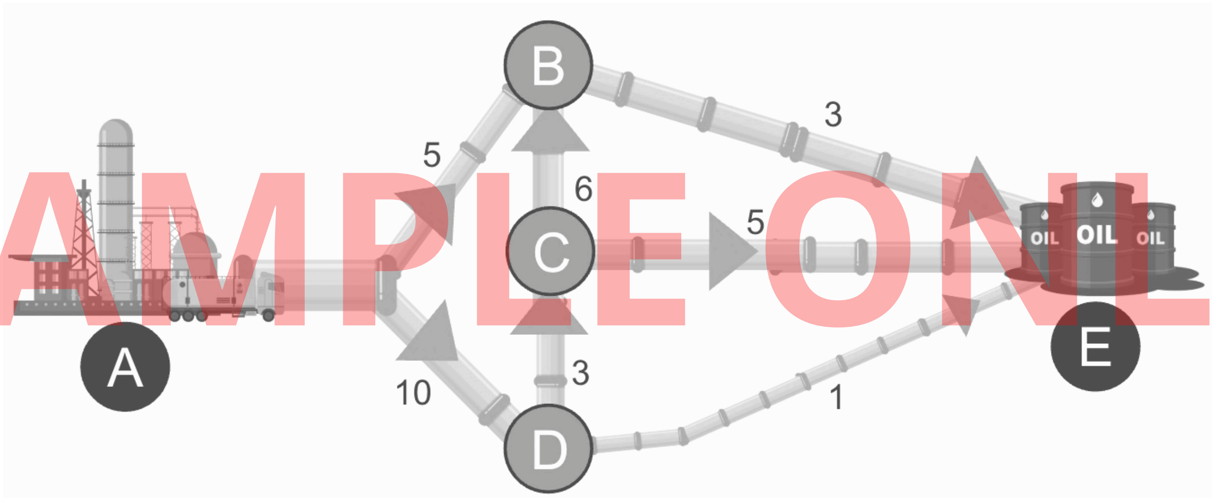
Minimum cut – Maximum flow theorem
In any flow network, the **maximum possible flow from the source to the sink is equal to the total capacity of the minimum cut** — the smallest total capacity of edges that, if removed, would completely separate the source from the sink.

The *weakest link* in the network — the narrowest set of pipes connecting the source side to the sink side — determines how much flow the whole system can carry.

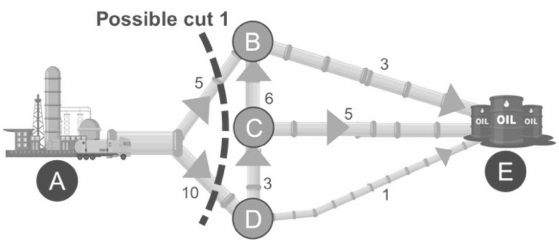
Steps to minimum cut – maximum flow theorem.
1. Cut the flow network in two – completely separating the source from the sink.
2. The **total capacity** of that cut tells you how much flow would be blocked by cutting those edges.
3. There are often multiple possible cuts. The cut with the smallest possible capacity (the **minimum cut**) results in the **maximum possible flow** that the network could carry.

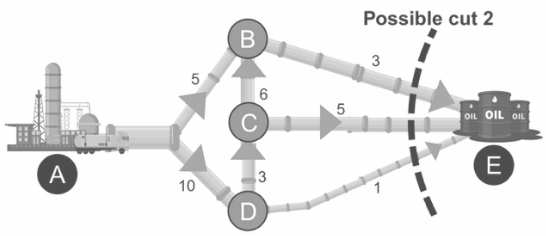
Worked example.

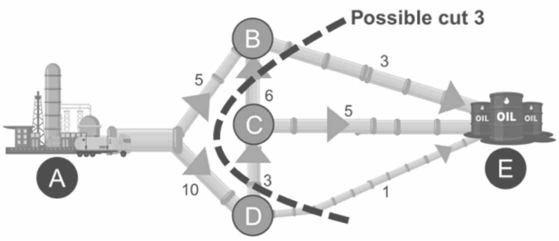
Determine the maximum volume of oil that can flow through the network of pipes from an oil processing plant (A) to storage (E). Measurements in litres per second.



Step 1: Find all the places where we could cut the network that would completely stop the flow of oil from the source (A) to the sink (E).







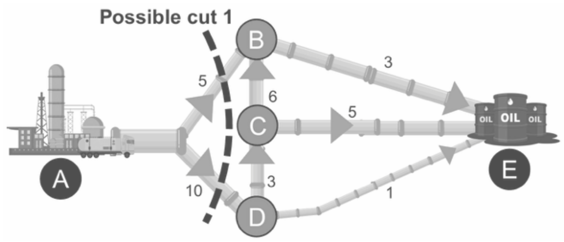
A **valid cut** is one that isolates the source from the sink.

Worked example continued.

Step 2. Find the capacity of each cut.

(how much oil would be blocked if that cut were made)

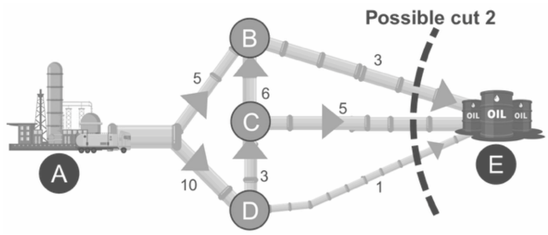
Note: Not all edges are counted, only the ones that would stop the flow of oil.



Cut through A → D = 10

Cut through A → B = 5

Total capacity of cut 1 = 15



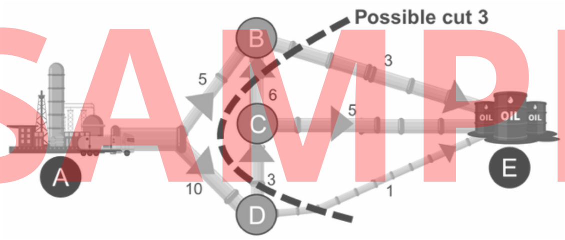
Cut through B → E = 3

Cut through C → E = 5

Cut through D → E = 1

Total capacity of cut 2 = 9

Cuts are only counted if they stop the flow from the source to the sink.



Cut through B → E = 3

Cut through D → C = 3

Cut through C → B = 0 (the oil was already cut off at D → C)

Cut through D → E = 1

Total capacity of cut 3 = 7 ← minimum cut

The **minimum cut** (cut #3) represents the **weakest choke point in the whole system** — the *smallest total capacity* of pipes that, if blocked, would completely stop oil from getting from A (source) to E (sink).

- If you were an engineer trying to increase total flow, the pipes involved in cut #3 are the pipes you should upgrade.
- If you were trying to shut the system off, the pipes in cut #3 are the fewest pipes you would need to close to completely stop all oil reaching E.

What the minimum cut tells us.

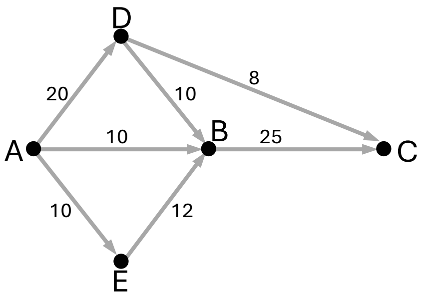
7 litres per second is the true maximum possible flow from A to E — the entire network cannot deliver more than that.

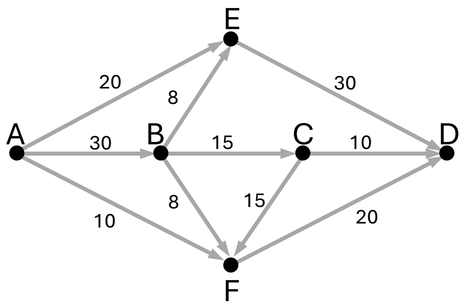
The edges on that minimum cut are the weakest links — they're the ones limiting how much oil can flow through the whole system.

This example is based on the example shown in a YouTube video by *Juddy Productions* called "*Minimum cuts and maximum flow rate*". Check it out if you are having trouble with this concept.

With your teacher.

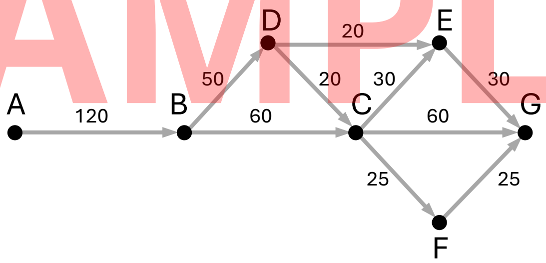
1. Find the maximum flow from source to sink in the following networks.





2. A highway network is represented by the graph below. Weights represent the capacity of different sections of the highway in cars per minute.

   a. Find the maximum flow of traffic from A to G.

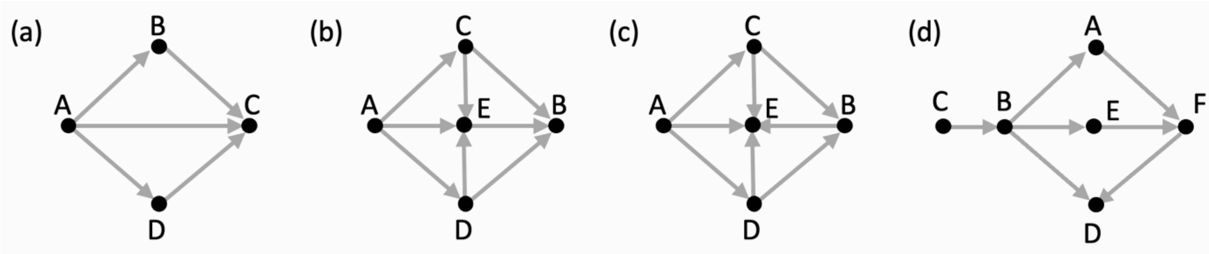


   b. Authorities would like to improve traffic flow but wish to improve only one section of road. Which sections will bring about improvement?

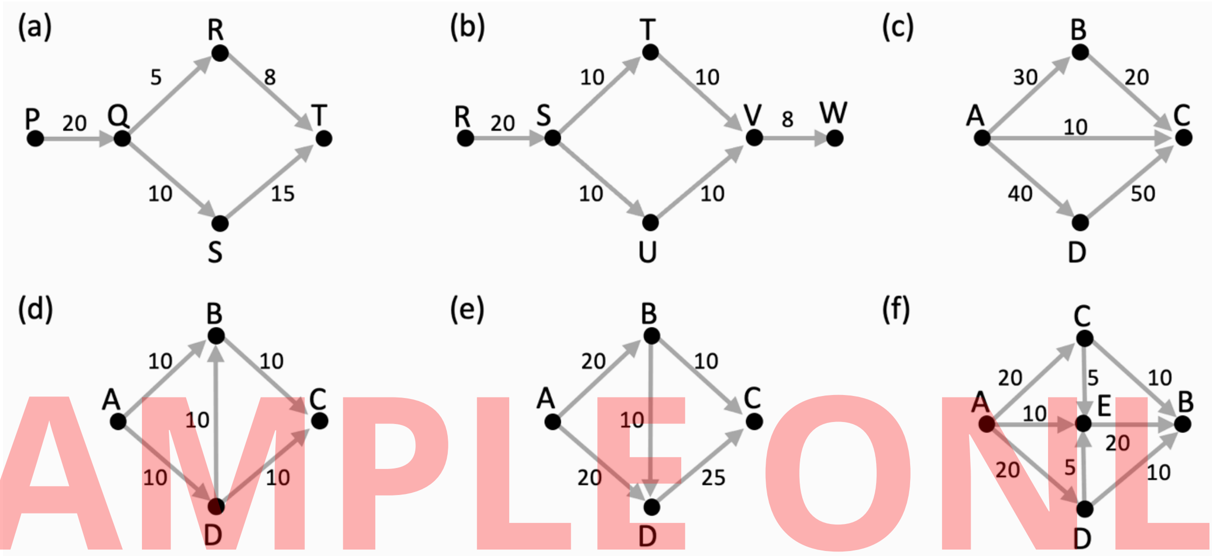
   c. By how much would the overall flow improve if the section DE was to be upgraded to 60 cars/min?

Try it yourself (FLOW) *Ans pg80*

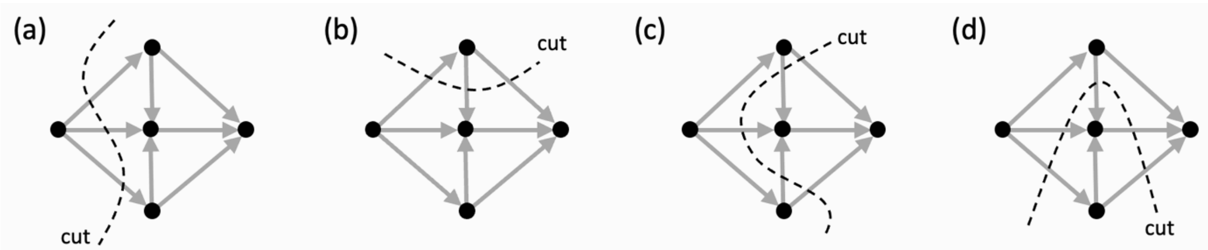
1. Identify the source and the sink in the following networks.



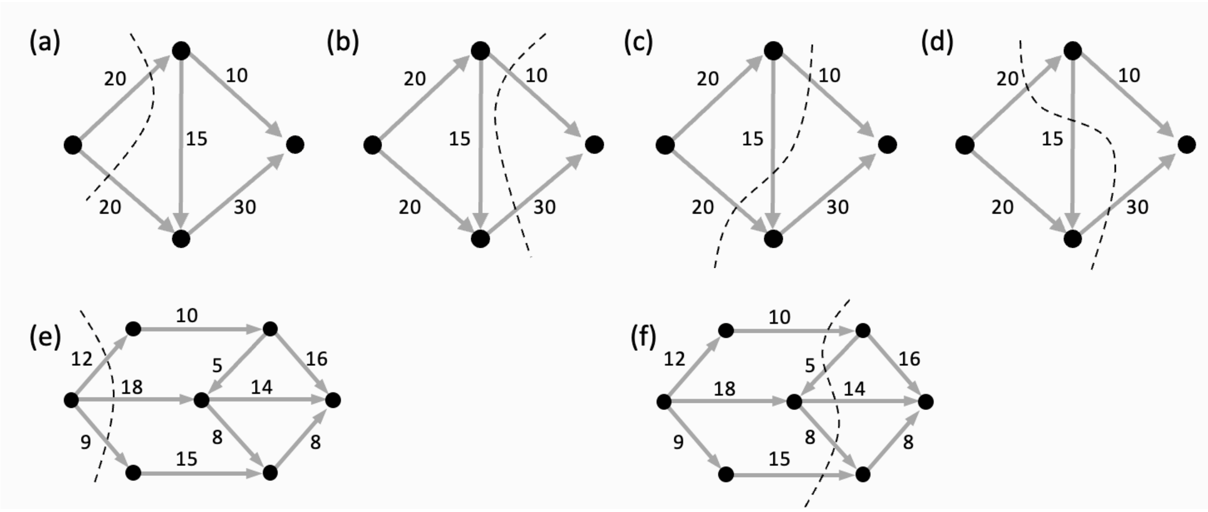
2. Find the maximum flow in each of the following networks 'by inspection'



3. A network cut is valid if it separates source and sink. Which of the following network cuts are valid?



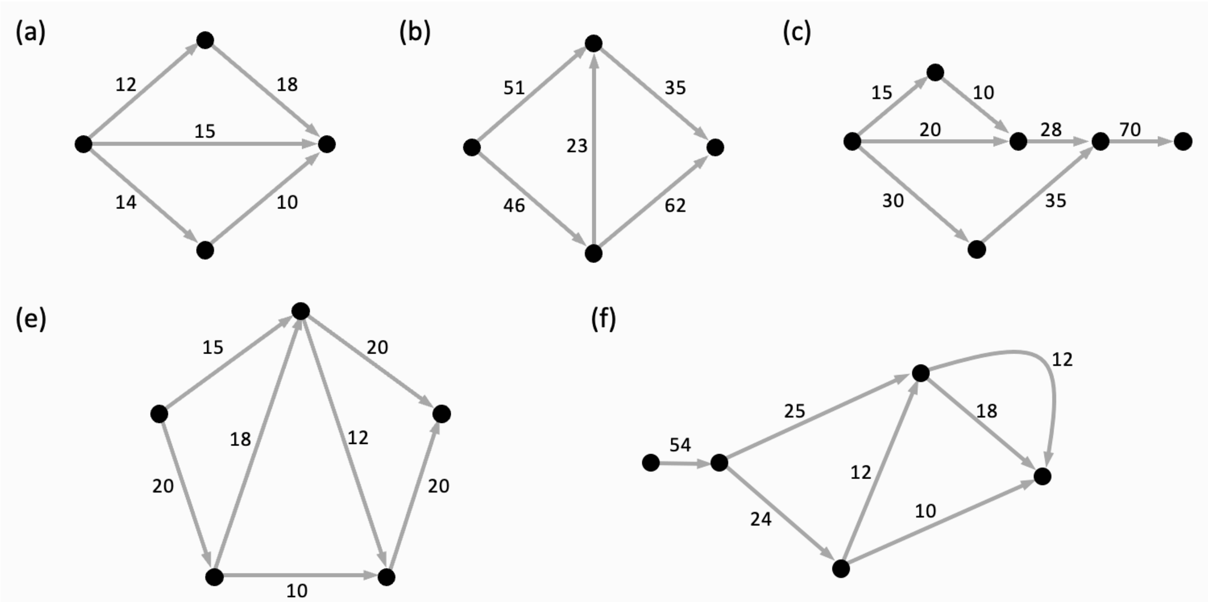
4. Calculate the capacity of each of the following network cuts.



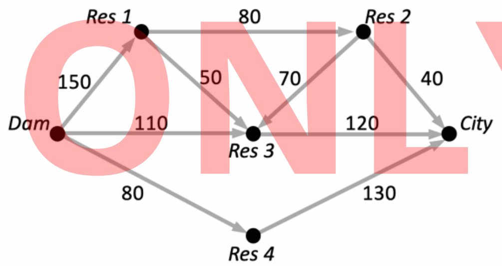
Try it yourself (FLOW cont) *Ans pg80*

Ans pg80

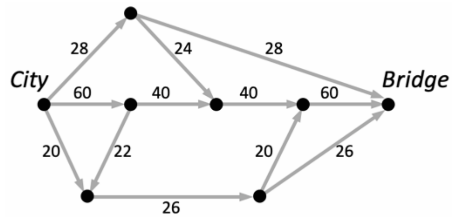
5. Use 'min cut = max flow' to calculate the maximum flow of each of the following networks.



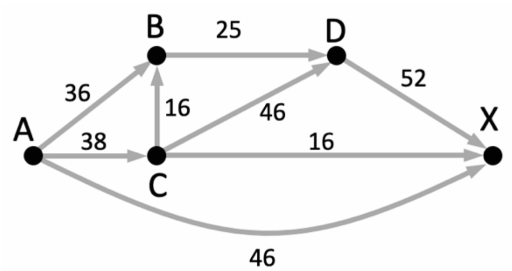
6. A network of pipes and reservoirs reticulate fresh water to a small city. The weights on the graph give the capacities of the pipes in ML/h. What is the maximum flow of water to the city?



7. The graph presents the major roads between the centre of a city and the city's bridge. Weights detail the capacities of each road in cars per minute. What is the maximum flow of the network?

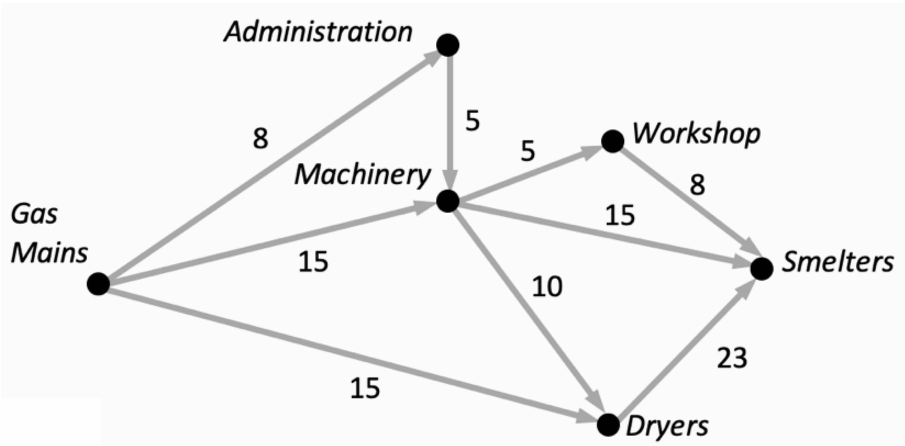


8. The graph shows the number of tonnes of air-freight that can be sent every day via different routes between two capital cities.
   a. What is the maximum flow of air freight from city A to city X?
   b. How could the overall capacity be increased by increasing the freight capacity on a single route. List some alternatives.

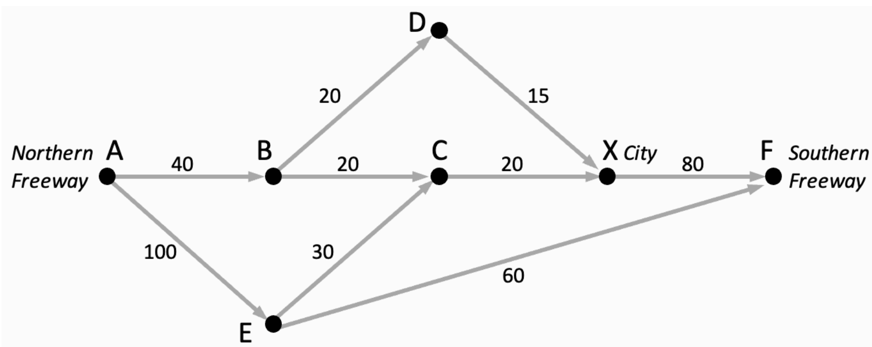


Try it yourself (FLOW cont) *Ans pg80*

9. Natural gas is pumped from a gas mains to different parts of a large factory according to the graph. Most gas is used in the smelters.
   a. What is the maximum capacity of the network to supply the smelters?
   b. In order to supply more gas to the smelters it is planned to upgrade the line between the mains and 'Machinery'. What is the maximum worthwhile upgade to this line?

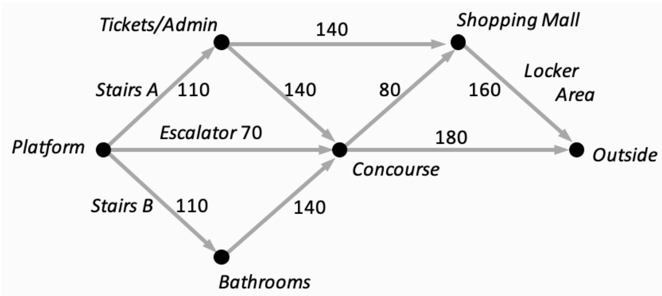


10. The graph shows how freeway traffic flows through and around a small city. (Weights are measured in cars per minute)
    a. Find maximum flow across the network from freeway to freeway.
    b. Which roads (considered individually) could be upgraded to improve flow?
    c. The city council is considering upgrading the road from D to X. What would be the maximum worthwhile upgrade and what would be the improved capacity of the system in this case?
    d. If, instead, the council was to upgrade the road from E to F; what would be the maximum worthwhile upgrade and what would be the improved capacity of the system?



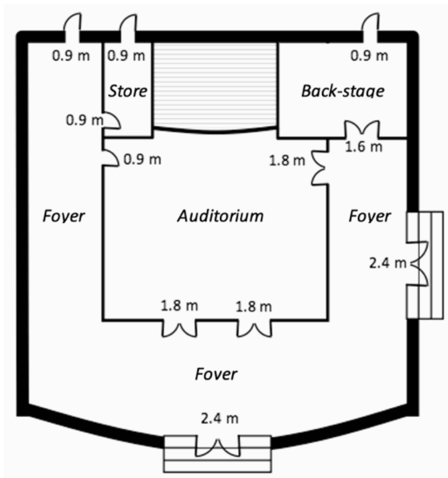
Try it yourself (FLOW cont) *Ans pg80*

11. The graph shows the capacity of routes to exit a subway train station. (Weights are measured in people per minute). Authorities wish to upgrade the capacity of the station to deal with an increasing number of passengers.
    a. What is the maximum flow of people from the platform to the 'outside'.
    b. Which edges on the graph (considered singularly) would, if upgraded, result in an overall improvement of flow?



    c. Find the maximum worthwhile upgrade to the escalator and the resulting maximum flow of the network.
    d. It has been suggested that moving the lockers to the corridor between the concourse and the shopping mall would improve the flow of their current corridor by 20 (ppm) but decrease the flow in their new position by 20 (ppm). Would this be worthwhile given that the escalator has been improved by your recommended amount?

12. The rate at which people can exit a building depends upon the width of its doors. The capacity of a door with regard to the throughput of people can be found using the formula: $C = 40\ w^2$ (people per minute) where w is the width of the door in m.
    a. Find the capacity of each of the doors of the building in the diagram.
    b. Draw an access network for the building weighted with the capacity of each door.



    c. Find the maximum flow of people in the event of an evacuation.
    d. How long would it take to evacuate if the auditorium holds 2500 people?
    e. Which of the following renovations would speed the time of an evacuation?
        - Make the foyer larger in size.
        - Widen the front and side external entrance doors.
        - An additional set of doors joining the auditorium to the foyer.
        - Make the door from 'back-stage' to outside a double door.

GM3 Networks

2026 version

Jess Bertram

BERTRAM EDUCATION